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INTERSECTIONS OF RANDOM WALKS WITH RANDOM SETS†

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ABSTRACT

Two general theorems about the intersections of a random walk with a random set are proved. The result is applied to the cases when the random set is a (deterministic) half-line and a two-sided random walk.

1. Introduction

Let $R(n)$, $n \in \mathbf{Z}_+$, denote a random walk taking values in $\mathbf{Z}^d$ as in Spitzer [10], with killing rate $\beta \in (0, 1)$, starting at 0. Then it is well known that if

$$f = P_0\{R(n) \neq 0, n > 0\} \quad \text{and} \quad g = E_0\left(\sum_{n=0}^{\infty} I\{R(n) = 0\}\right),$$

then $fg = 1$. To see this we need only consider the last hitting time σ,

$$\sigma = \sup\{n : R(n) = 0\},$$

and note that

$$P\{\sigma = n\} = P\{R(n) = 0; R(j) \neq 0, j > n\}$$

$$= P\{R(n) = 0\} f,$$

and hence

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$$1=\sum_{n=0}^{\infty} P\{\sigma=n\}$$

$$=f\sum_{n=0}^{\infty} P\{R(n)=0\}=fg.$$

If we take a slightly more general case where $A\subset \mathbf{Z}^d$ is a finite set and we let

$$f_x=P_x\{R(n)\notin A,\, n>0\} \quad\text{and}\quad g_x=E_0\left(\sum_{n=0}^{\infty} I\{R(n)=x\}\right),$$

then by considering $\sigma=\sup\{n: R(n)\in A\}$ we get, if $0\in A$,

$$1=\sum_{x\in A} f_x g_x. \tag{1.1}$$

In this paper we generalize those ideas to the case where A is random. If A is a fixed finite set with m points and we choose one of the sets $A-x=\{y-x: y\in A\}$, with $x\in A$ at random with probability $1/m$ for each translate (1.1) gives

$$E(f_0G)=1, \tag{1.2}$$

where $G=\Sigma_{x\in A}\, g_x$ (here the E refers to expectation with respect to the uniform probability measure on $\Omega=\{A-x: x\in A\}$). If we consider more general measures on the set A we can get further generalizations of (1.2).

One motivation for this lies in the study of intersections of random walks. If $R_1(n)$ and $R_2(n)$ are independent simple random walks starting at 0, one is interested in estimating

$$f(n)=P\{R_1(i)\neq R_2(j),\, 0<i\leqq n,\, 0<j\leqq n\}.$$

We might hope to solve this kind of problem in this context, using the random set $A=\{R_1(i), 0<i\leqq n\}$ and $R=R_2$. Unfortunately this does not seem to work. The problem that arises is that the measure on A is not translation invariant, i.e., A and $A-x$ do not have the same probability for $x\in A$.

The ideas of (1.2) do apply to the random walk case if we choose A to be the path of a "two-sided" random walk $R_1(n)$, $-\infty<n<\infty$, with killing rate β in each direction. In this paper we derive a generalization of (1.2), Theorem 2.2, assuming essentially only that the measure on A be translation invariant and symmetric about 0, and that R is a symmetric random walk.

As an application of the result we give a proof of the following: let R_1, R_2, R_3 be independent simple random walks and

$$F(n) = P\{R_1(i) \neq R_3(j), R_2(i) \neq R_3(j), 0 \leqq i \leqq n, 0 < j \leqq n\},$$

then there exist constants $0 < c_1 < c_2 < \infty$ such that

$$(1.3) \qquad \begin{cases} c_1 n^{(d/2)-2} \leqq F(n) \leqq c_2 n^{(d/2)-2}, & d = 1, 2, 3, \\ c_1(\log n)^{-1} \leqq F(n) \leqq c_2(\log n)^{-1}, & d = 4, \\ c_1 \leqq F(n), & d \geqq 5. \end{cases}$$

We actually will give a proof only in the cases $d = 2, 3$. The case $d = 1$ can be handled easily using methods from Chapter 3 of Feller [5]. The $d \geqq 5$ follows from the fact that simple random walks intersect only a finite number of times (see, for example, Erdös and Taylor [3]); and the $d = 4$ case was proved in Lawler [7] using *ad hoc* methods very similar to those in this paper. The cases $d = 2, 3$ are also "well known", and a similar but not identical result follows from the work of Felder and Fröhlich [4], but no proof of (1.3) seems to be in print. From (1.3) one easily gets (see Lawler [9])

$$c_1 n^{(d/2)-2} \leqq f(n) \leqq \sqrt{c_2}\, n^{(d/4)-1}, \qquad d = 1, 2, 3,$$

$$c_1(\log n)^{-1} \leqq f(n) \leqq \sqrt{c_2}(\log n)^{-1/2}, \qquad d = 4.$$

For $d = 4$ it was shown in Lawler [8] that the right inequality is nearly sharp, i.e. that

$$\lim_{n \to \infty} \frac{\log f(n)}{\log \log n} = -\tfrac{1}{2}.$$

For $d = 1$ it can be shown that $f(n) \sim c_3 n^{-1}$ so that neither inequality is sharp. It is believed for heuristic reasons (see Duplantier [2]) that the inequality is also not sharp for $d < 4$. In fact, for $d = 2$, it has recently been proved by Burdzy and Lawler [1] for some $\varepsilon > 0$

$$-\tfrac{3}{4} \leqq \liminf_{n \to \infty} \frac{\log f(n)}{\log n}$$

$$\leqq \limsup_{n \to \infty} \frac{\log f(n)}{\log n} \leqq -\tfrac{1}{2} - \varepsilon,$$

i.e. the inequality is not sharp.

The proof of (1.3) requires a little more work than Theorem 2.2. From Theorem 2.2, using the notation of (1.2) we get essentially

$$E(FG) \approx 1.$$

It is also routine to show

$$E(G) \approx n^{2-(d/2)}, \qquad d = 2, 3,$$

but care is needed to conclude from these facts that

$$E(F) \approx [E(G)]^{-1}.$$

In this paper we also derive another identity, Theorem 2.1, by a similar argument. As an application of this we derive an estimate for the probability that a simple random walk with killing rate $\beta > 0$ avoids a half line. This is closely related to a recent estimate of Kesten [6] on the harmonic measure of a segment in $\mathbf{Z}^2$ and in fact we could derive estimate (i) of that paper from our estimate.

Throughout this paper we will use $0 < c_1 < c_2 < \infty$ to represent constants, independent of everything except dimension, which may vary from line to line.

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2. Main theorems

We let $A(n)$ be a random two-sided sequence of points in $\mathbf{Z}^d \cup \{\infty\}$ which is invariant under translations and time reversal. More specifically, $A(n)$, $n \in \mathbf{Z}$, take values in $\mathbf{Z}^d \cup \{\infty\}$ and satisfy

(2.1) $$A(0) = 0;$$

(2.2) $$\text{if } n > 0 \text{ and } A(n) = \infty, \text{ then } A(m) = \infty \text{ for } m \geqq n;$$

(2.3) $$\text{if } n < 0 \text{ and } A(n) = \infty, \text{ then } A(m) = \infty \text{ for } m \leqq n;$$

if $x_{-m}, \ldots, x_0, \ldots, x_n$ is any sequence of points in $\mathbf{Z}^d \cup \{\infty\}$,

$$P\{A(-m) = x_{-m}, A(-m+1) = x_{-m+1}, \ldots, A(n-1) = x_{n-1}, A(n) = x_n\}$$

$$= P\{A(-n) = x_n, A(-n+1) = x_{n-1}, \ldots, A(m-1) = x_{-m+1}, A(m) = x_{-m}\}$$
(2.4)

and if $-m \leqq k \leqq n$ with $x_k \neq \infty$, $x_0 = 0$,

$$P\{A(-m)=x_{-m}, A(-m+1)=x_{-m+1}, \ldots, A(n-1)=x_{n-1}, A(n)=x_n\}$$

$$(2.5) \qquad = P\{A(-m-k)=y_{-m}, A(-m-k+1) = y_{-m+1}, \ldots, A(n-k-1)=y_{n-1}, A(n-k)=y_n\}$$

where

$$y_j = \begin{cases} x_j - x_k, & x_j \neq \infty, \\ \infty, & x_j = \infty. \end{cases}$$

By a random walk $R(n)$, $n \geqq 0$, we will mean a symmetric walk with killing rate $\beta \in [0, 1)$, taking values in $\mathbf{Z}^d \cup \{\overline{\infty}\}$, i.e.

$$(2.6) \qquad R(0)=0,$$

$$(2.7) \qquad P\{R(n+1)=\overline{\infty} \mid R(n)=\overline{\infty}\}=1,$$

$$(2.8) \qquad P\{R(n+1)=\overline{\infty} \mid R(n)\neq\overline{\infty}\}=\beta,$$

$$(2.9) \qquad P\{R(n+1)=x \mid R(n)=y\}=(1-\beta)\mu(x-y),$$

where μ is a probability measure on $\mathbf{Z}^d$ satisfying $\mu(x)=\mu(-x)$. Here $\overline{\infty}$ is a "cemetery point" assumed to be different from the ∞ in the definition of A. If R^1, R^2 are independent random walks with the same distribution (i.e. same β and μ) and we define

$$A_R(n) = \begin{cases} R^1(n), & n \geqq 0, \\ R^2(-n), & n \leqq 0, \end{cases}$$

then $A_R(n)$ satisfies (2.1)–(2.5). However, not every $A(n)$ satisfying (2.1)–(2.5) comes from such a random walk.

We now assume we are given $A(n)$ satisfying (2.1)–(2.5) on a probability space (Ω_1, P_1) and $R(n)$ satisfying (2.6)–(2.9) on a different space (Ω_2, P_2). Let $(\Omega, P)=(\Omega_1 \times \Omega_2, P_1 \times P_2)$. For any set $A \subset \mathbf{Z}^d$, we let

$$\mathrm{Es}(A) = P_2\{\omega_2 : R(j, \omega_2) \notin A, 1 \leqq j < \infty\}.$$

We define a number of sets and random variables on Ω^1, Ω:

$$J_n^+(\omega_1) = \mathrm{Es}[\{A(k, \omega_1) : k \geqq n\}],$$

$$J_n^-(\omega_1) = \mathrm{Es}[\{A(k, \omega_1) : k \leqq n\}],$$

$$J(\omega_1) = J_\infty^-(\omega_1) = J_{-\infty}^+(\omega_1),$$

$$B_n^+ = \{(\omega_1, \omega_2) : R(j, \omega_2) \neq A(k, \omega_1), n \leqq k < \infty, 1 \leqq j < \infty\},$$

$$D^+(\omega_1) = \text{indicator function of the set } \{A(k,\omega_1) \neq 0, 0 < k < \infty\},$$

$$D^-(\omega_1) = \text{indicator function of the set } \{A(k,\omega_1) \neq 0, -\infty < k < 0\},$$

$$\xi_n(\omega_1,\omega_2) = \inf\{j \geqq 1 : R(j,\omega_2) = A(n,\omega_1)\},$$

$$\tau(\omega_1,\omega_2) = \inf\{n \in \mathbf{Z} : \xi_n(\omega_1,\omega_2) < \infty\}.$$

We now state the main results in terms of the above random variables. We use E_1 to denote expectations with respect to P_1.

THEOREM 2.1. *Suppose A is transient, i.e.*

$$P_1\{\exists x \in \mathbf{Z}^d \text{ with } A(n) = x \text{ for infinitely many } n\} = 0.$$

Then

$$E_1(D^+J) = E_1(D^+J_0^+J_1^+).$$

THEOREM 2.2. *For $x \in \mathbf{Z}^d$ let*

$$g(x) = \sum_{j=0}^{\infty} P_2\{R(j,\omega_2) = x\} \quad \text{and} \quad G(\omega_1) = \sum_{m=-\infty}^{\infty} g(A(m,\omega_1)),$$

where $g(\infty) = 0$. Then, if $E_1(G) < \infty$,

$$E_1(GD^+J) = 1.$$

From the theorems we get the immediate corollary:

COROLLARY 2.3. *Suppose $P_1\{A(n) = 0$ for some $n \neq 0\} = 0$. Then*
(a) $E_1(J) = E_1(J_0^+ J_1^+)$;
(b) *if $E_1(G) < \infty$, then $E_1(GJ) = 1$.*

Note that by translation invariance $P_1\{A(n) = 0$ for some $n \neq 0\} = 0$ if and only if A has no double points, i.e.

$$P_1\{A(n) \neq A(m) \text{ for all } n < m, A(n) \neq \infty\} = 1.$$

Before proving these theorems, we do an example to show how they can be used. Suppose $A(n)$ is not random but just a line

$$A(n) = (n, 0, 0, \ldots, 0).$$

Actually, this does not satisfy (2.4); however, we may instead suppose that with probability $\frac{1}{2}$, $A(n) = (n, 0, 0, \ldots, 0)$, $n \in \mathbf{Z}$, and with probability $\frac{1}{2}$, $A(n) =$

$(-n, 0, 0, \ldots, 0)$, $n \in \mathbf{Z}$. Let $R(j)$ be a simple random walk in $\mathbf{Z}^d$ ($d \geqq 2$) with killing rate $\beta \in [0, 1)$. Then J, J_0^+, J_1^+, G are not random. Corollary 2.3(a) gives

$$\begin{aligned} &P_2\{R(j) \neq A(n), j \geqq 1, \ -\infty < n < \infty\} \\ (2.10) \qquad &\quad = P_2\{R(j) \neq A(n), j \geqq 1, n \geqq 0\} P_2\{R(j) \neq A(n), j \geqq 1, n < 0\}. \end{aligned}$$

If $\beta = 0$ and $d = 2, 3$ both sides are zero. Otherwise we get a surprising fact: if

$$V^+ = \{\omega_2 : R(j, \omega_2) \neq A(n), n \geqq 0, j \geqq 1\}$$

and

$$V^- = \{\omega_2 : R(j, \omega_2) \neq A(n), n < 0, j \geqq 1\},$$

then V^+ and V^- are independent events! Corollary 3.2(b) gives

$$P_2\{R(j) \neq A(n), j \geqq 1, \ -\infty < n < \infty\} = \left[\sum_{j=0}^{\infty} \sum_{n=-\infty}^{\infty} P\{R(j) = A(n)\} \right]^{-1}$$

(this can be proved other ways). In particular by doing standard estimates on the RHS for $d = 2, 3$ we can get asymptotic expressions as $\beta \to 0$,

$$(2.11) \qquad P_2\{R(j) \neq A(n), j \geqq 1, \ -\infty < n < \infty\} \sim \begin{cases} c\sqrt{\beta}, & d = 2, \\ \dfrac{c}{|\log \beta|}, & d = 3. \end{cases}$$

Below we will show that

$$\begin{aligned} &P_2\{R(j) \neq A(n), j \geqq 1, 1 \leqq n < \infty\} \\ (2.12) \qquad &\quad \leqq 2d P_2\{R(j) \neq A(n), j \geqq 1, 0 \leqq n < \infty\}. \end{aligned}$$

This combined with (2.10) and (2.11) then gives:

COROLLARY 2.4. *There exist constants* $0 < c_1 < c_2 < \infty$ *such that if* $R(n)$ *is a simple random walk in* $\mathbf{Z}^d$, $d = 2, 3$, *with killing rate* $\beta > 0$, *defined on a probability space* (Ω_2, P_2) *and* A *is the half-line*

$$A = \{(n, 0) : n \geqq 0\}, \qquad d = 2,$$

$$A = \{(n, 0, 0) : n \geqq 0\}, \qquad d = 3,$$

then

$$c_1 \beta^{1/4} \leqq P_2\{R(j) \notin A, j \geqq 1\} \leqq c_2 \beta^{1/4}, \qquad d = 2,$$

$$\frac{c_1}{\sqrt{|\log \beta|}} \leqq P_2\{R(j) \notin A, j \geqq 1\} \leqq \frac{c_2}{\sqrt{|\log \beta|}}, \qquad d = 3.$$

This result is connected with the discrete harmonic measure of the endpoint of a line and with a little more work we could prove estimate (i) in Kesten [6]. To prove (2.12), let $\sigma_1 = \inf\{j \geqq 1 : R(j) = 0\}$ and, for $i > 1$, $\sigma_i = \inf\{j > \sigma_{i-1} : R(j) = 0\}$. Let

$$\Delta = \sup\{i : \sigma_i < \infty\}.$$

Then

$$\begin{aligned} &P\{\Delta = i, R(j) \neq A(n), j \geqq 1, 0 < n < \infty\} \\ &\quad = P\{\sigma_i < \infty, R(j) \neq A(n), 1 \leqq j \leqq \sigma_i, 0 < n < \infty\} \\ &\quad\quad \cdot P\{R(j) \neq A(n), 0 < j < \infty, 0 \leqq n < \infty\}. \end{aligned}$$

It is easy to see that

$$P\{\sigma_1 < \infty, R(j) \neq A(n), 1 \leqq j \leqq \sigma_1, 0 < n < \infty\} \leqq \frac{2d-1}{2d}.$$

Similarly,

$$P\{\sigma_i < \infty, R(j) \neq A(n), 1 \leqq j \leqq \sigma_i, 0 < n < \infty\} \leqq \left(\frac{2d-1}{2d}\right)^i.$$

Therefore,

$$\begin{aligned} &P\{R(j) \neq A(n), j \geqq 1, 0 < n < \infty\} \\ &\quad = \sum_{i=0}^{\infty} P\{\Delta = i, R(j) \neq A(n), j \geqq 1, 0 < n < \infty\} \\ &\quad \leqq \sum_{i=0}^{\infty} \left(\frac{2d-1}{2d}\right)^i P\{R(j) \neq A(n), j \geqq 1, 0 \leqq n < \infty\} \\ &\quad = 2d\, P\{R(j) \neq A(n), j \geqq 1, 0 \leqq n < \infty\}. \end{aligned}$$

Proof of Theorem 2.1. We will consider $E_1(D^-(J_0^- - J))$. Note that by (2.4), $E_1(D^-J_0^-) = E_1(D^+J_0^+)$, $E_1(D^-J) = E_1(D^+J)$. Let

$$C_n = \{(\omega_1, \omega_2) : \tau(\omega_1, \omega_2) = n, D^-(\omega_1) = 1\}.$$

Then

$$E_1(D^-(J_0^- - J)) = \sum_{n=1}^{\infty} P(C_n). \tag{2.13}$$

Let $C_{n,j} = \{(\omega_1, \omega_2) \in C_n : \xi_n(\omega_1, \omega_2) = j\}$. Then for $n, j \geqq 1$, $(\omega_1, \omega_2) \in C_{n,j}$ if and only if

(a) $A(m, \omega_1) \neq 0, \ -\infty < m < 0$,

(b) $R(j, \omega_2) = A(n, \omega_1)$,

(c) $R(k, \omega_2) \neq A(n, \omega_1), \ 1 \leqq k < j$,

(d) $R(k, \omega_2) \neq A(m, \omega_1), \ 1 \leqq k \leqq j, \ -\infty < m < n$,

(e) $R(k, \omega_2) \neq A(m, \omega_1), j < k < \infty, \ -\infty < m < n$.

For fixed j, n we write

$$R^1(k, \omega_2) = R(j - k, \omega_2) - R(j, \omega_2), \qquad 0 \leqq k \leqq j,$$

$$R^2(k, \omega_2) = R(j + k, \omega_2) - R(j, \omega_2), \qquad 0 \leqq k < \infty,$$

and

$$\bar{A}(m, \omega_1) = A(n - m, \omega_1) - A(n, \omega_1), \qquad -\infty < m < \infty.$$

(In these definitions, we use the rules $\infty = -\infty = x + \infty = x - \infty$ and similarly for $\overline{\infty}$.)

Again suppose $(\omega_1, \omega_2) \in C_{n,j}$. Then (b) and (d) imply that $A(m, \omega_1) \neq A(n, \omega_1)$ for $-\infty < m < n$ or

(a)′ $\bar{A}(m, \omega_1) \neq 0, 0 < m < \infty$.

Similarly (b)–(e) give

(b)′ $R^1(j, \omega_2) = \bar{A}(n, \omega_1)$,

(c)′ $R^1(k, \omega_2) \neq \bar{A}(m, \omega_1), \ 1 \leqq k < j, 0 \leqq m \leqq n$,

(d)′ $R^1(k, \omega_2) \neq \bar{A}(m, \omega_1), \ 1 \leqq k \leqq j, n < m < \infty$,

(e)′ $R^2(k, \omega_2) \neq \bar{A}(m, \omega_1), 0 < k < \infty, 0 < m < \infty$.

Moreover, $(\omega_1, \omega_2) \in C_{n,j}$ if and only if (a)′–(e)′ hold: hence

$$P(C_{n,j}) = P\{\text{(a)′–(e)′ hold}\}.$$

For fixed n, j, (2.1)–(2.5) say that $\bar{A}$ has the same distribution as A and (2.6)–(2.9) say that R^1, R^2 are independent random walks with the same distribution as R. (We note that the latter claim uses the fact that for $(\omega_1, \omega_2) \in C_{n,j}, R(j, \omega_2) \neq \overline{\infty}$.) Therefore $P(C_{n,j}) = P(\tilde{C}_{n,j})$ where $\tilde{C}_{n,j}$ is the set of (ω_1, ω_2) satisfying

(a)″ $A(m, \omega_1) \neq 0, 0 < m < \infty$,

(b)″ $R^1(j, \omega_2) = A(n, \omega_1)$,

(c)″ $R^1(k, \omega_2) \neq A(m, \omega_1), \ 1 \leqq k < j, 0 \leqq m \leqq n$,

(d)″ $R^1(k, \omega_2) \neq A(m, \omega_1), \ 1 \leqq k \leqq j, n < m < \infty$,

(e)″ $R^2(k, \omega_2) \neq A(m, \omega_1), 0 < k < \infty, 0 < m < \infty$.

Let

$$\eta(\omega_1,\omega_2)=\inf\{j \geqq 1 : R^1(j,\omega_2)=A(n,\omega_1) \text{ for some } n \geqq 0\},$$

$$\sigma(\omega_1,\omega_2)=\sup\{n : R^1(\eta,\omega_2)=A(n,\omega_1)\}.$$

Then $\{(\mathrm{b})''\text{–}(\mathrm{d})''\}=\{\eta=j,\sigma=n\}$. Since R^2 is independent of R^1, we get

$$P(\tilde{C}_{n,j})=E_1(D^+J_1^+I\{\eta=j,\sigma=n\}),$$

where I denotes indicator function. Summing over n and j we get

$$\sum_{n=1}^{\infty}\sum_{j=1}^{\infty}P(\tilde{C}_{n,j})=E_1\left(D^+J_1^+\sum_{n=1}^{\infty}\tilde{I}\{\sigma=n\}\right),$$

where $\tilde{I}\{\sigma=n\}=E(I\{\sigma=n\}\mid\omega_1)$. But

$$\sum_{n=0}^{\infty}\tilde{I}\{\sigma=n\}+J_0^++\tilde{I}\{\sigma=\infty\}=1.$$

Since A is transient, $P\{\sigma=\infty\}=0$, therefore

$$\sum_{n=0}^{\infty}\tilde{I}\{\sigma=n\}+J_0^+=1 \qquad \text{a.s.}$$

Hence by (2.13)

$$E_1(D^-(J_0^- - J))=E_1(D^+J_1^+(1-\tilde{I}\{\sigma=0\}-J_0^+)).$$

For fixed ω_1 with $D^+(\omega_1)=1$, standard Markov time arguments give

$$J_1^+(\omega_1)=J_0^+(\omega_1)+\tilde{I}\{\sigma=0\}(\omega_1)J_1^+(\omega_1).$$

Hence

$$E_1(D^-J_0^-)-E_1(D^-J)=E_1(D^+J_0^+-D^+J_0^+J_1^+).$$

But as mentioned before, $E_1(D^-J_0^-)=E_1(D^+J_0^+)$ and $E_1(D^-J)=E_1(D^+J)$. Therefore

$$E_1(D^+J)=E_1(D^+J_0^+J_1^+).$$

PROOF OF THEOREM 2.2. Let

$$\eta(\omega_1,\omega_2)=\sup\{j\geqq 0 : R(j,\omega_2)=A(n,\omega_1) \text{ for some } -\infty<n<\infty\},$$

$$\sigma(\omega_1,\omega_2)=\sup\{n\geqq 0 : R(\eta,\omega_2)=A(n,\omega_1)\}.$$

We first note that $P\{\eta<\infty,|\sigma|<\infty\}=1$. (Assume not, then if

$$H(\omega_1, \omega_2) = \sum_{n=-\infty}^{\infty} \sum_{j=0}^{\infty} I\{R(j, \omega_2) = A(n, \omega_1)\},$$

$P\{H = \infty\} > 0$ and hence $E(H) = \infty$. But $E(H) = E_1(G) < \infty$.) Therefore

$$1 = \sum_{j=0}^{\infty} \sum_{n=-\infty}^{\infty} P\{\eta = j, \sigma = n\}.$$

Let $B_{n,j} = \{\eta = j, \sigma = n\}$. Then $(\omega_1, \omega_2) \in B_{n,j}$ if and only if

(a) $A(n, \omega_1) = R(j, \omega_2)$,

(b) $A(m, \omega_1) \neq A(n, \omega_1)$, $n < m < \infty$,

(c) $R(k, \omega_2) \neq A(m, \omega_1)$, $j < k < \infty$, $-\infty < m < \infty$.

For fixed n, j we define R^1, R^2, $\tilde{A}$ by

$$R^1(k, \omega_2) = R(j-k, \omega_2) - R(j, \omega_2), \qquad 0 \leqq k \leqq j,$$

$$R^2(k, \omega_2) = R(j+k, \omega_2) - R(j, \omega_2), \qquad 0 \leqq k < \infty,$$

$$\tilde{A}(m, \omega_1) = A(n+m, \omega_1) - A(n, \omega_1), \qquad -\infty < m < \infty,$$

with the same conventions about ∞ and $\overline{\infty}$ as in the previous proof. Then (a), (b), (c) are equivalent to

(a)′ $R^1(j, \omega_2) = \tilde{A}(-n, \omega_1)$,

(b)′ $\tilde{A}(m, \omega_1) \neq 0$, $0 < m < \infty$,

(c)′ $R^2(k, \omega_2) \neq \tilde{A}(m, \omega)$, $0 < k < \infty$, $-\infty < m < \infty$.

Again we use (2.1)–(2.9) to conclude that if $\tilde{B}_{n,j} = \{\text{(a)}''\text{–(c)}'' \text{ holds}\}$ where

(a)″ $R^1(j, \omega_2) = A(-n, \omega_1)$,

(b)″ $A(m, \omega_1) \neq 0$, $0 < m < \infty$,

(c)″ $R^2(k, \omega_2) \neq A(m, \omega_1)$, $0 < k < \infty$, $-\infty < m < \infty$,

then $P(\tilde{B}_{n,j}) = P(B_{n,j})$. Since R^1 and R^2 are independent we get

$$\begin{aligned} 1 &= \sum_{n=-\infty}^{\infty} \sum_{j=0}^{\infty} P\{\text{(a)}''\text{–(c)}'' \text{ hold}\} \\ &= E_1\left(D^+(\omega_1) J(\omega_1) \sum_{n=-\infty}^{\infty} \sum_{j=0}^{\infty} P_2\{R(j, \omega_2) = A(n, \omega_1)\}\right) \\ &= E_1(D^+ JG). \end{aligned}$$

The proof of Theorem 2.2 can be used to give a more general result which we will need. Let $\{x_n\}_{n=-\infty}^{\infty}$ be a sequence of points in $\mathbf{Z}^d \cup \{\infty\}$ with $x_0 = 0$. If $x_k \neq \infty$ we define

$$(T_k x)_n = \begin{cases} x_{n+k} - x_k, & x_{n+k} \neq \infty, \\ \infty, & x_{n+k} = \infty. \end{cases}$$

We call a function Φ on sequences invariant under translations if $\Phi(T_k x) = \Phi(x)$ whenever $x_k \neq \infty$. Then

COROLLARY 2.5. *If Φ is a bounded function on sequences, invariant under translations, and if $E_1(G) < \infty$, then*

$$E_1(GD^+ J\Phi(A)) = E_1(\Phi(A)).$$

PROOF. As in the above theorem, if $B_{n,j,a,b} = \{\eta = j, \sigma = n, a \leqq \Phi \leqq b\}$ we get

$$P\{a \leqq \Phi \leqq b\} = \sum_{j=0}^{\infty} \sum_{n=-\infty}^{\infty} P(B_{n,j,a,b}).$$

Since $\Phi(A) = \Phi(\tilde{A})$ we get

$$P(B_{n,j,a,b}) = P\{\text{(a)}''\text{–(c)}'' \text{ hold}, a \leqq \Phi \leqq b\},$$

and summing over all n, j gives

$$P\{a \leqq \Phi \leqq b\} = E_1(D^+ JG\, I(a \leqq \Phi \leqq b)).$$

Since this holds for every $a \leqq b$, the corollary follows.

3. Upper bound for $F(N)$

Let S^1, S^2, S^3 be independent simple random walks starting at the origin in $\mathbf{Z}^d$ and let

$$\Pi_i[0, N] = \{S^i(j) : 0 \leqq j \leqq N\}, \qquad \Pi_i(0, N] = \{S^i(j) : 0 < j \leqq N\}$$

and

$$F(N) = P\{\Pi_1(0, N] \cap (\Pi_2[0, N] \cup \Pi_3[0, N]) = \varnothing\}.$$

In the next two sections we prove:

THEOREM 3.1. *There exist constants $0 < c_1 < c_2 < \infty$ such that*

$$c_1 N^{-1/2} \leqq F(N) \leqq c_2 N^{-1/2}, \qquad d = 3,$$

$$c_1 N^{-1} \leqq F(N) \leqq c_2 N^{-1}, \qquad d = 2.$$

This section will be devoted to showing how the results of §2 can be used to prove the upper bound.

In the notation of the previous section we let $A(n)$ be a two-sided simple random walk with killing rate $\beta = \beta_N = 1/N$, and $R(n)$ a third walk independent of A also with killing rate β_N. In other words, if we let $\delta^1, \delta^2, \delta^3$ be random variables with

$$P\{\delta^i = k\} = \frac{1}{N}\left(1 - \frac{1}{N}\right)^k.$$

independent of S^1, S^2, S^3, as well as each other, we may choose

$$A(n) = \begin{cases} S^2(n), & 0 \leqq n \leqq \delta^2, \\ S^3(-n), & -\delta^3 \leqq n \leqq 0, \\ \infty, & \text{otherwise}, \end{cases} \quad \text{and} \quad R(n) = \begin{cases} S^1(n), & 0 \leqq n \leqq \delta^1, \\ \overline{\infty}, & \text{otherwise}, \end{cases}$$

and then $A(n)$ and $R(n)$ satisfy (2.1)–(2.9). We assume that S^1 is defined on (Ω_2, P_2); S^2, S^3 defined on (Ω_1, P_1); and $(\Omega, P) = (\Omega_1 \times \Omega_2, P_1 \times P_2)$ as in §2.

Let $\Phi(\omega_1) = \delta^2(\omega_1) + \delta^3(\omega_1) - 1$ and let K_1, K_2, K_3 be the indicator functions of the events

$$\left\{\frac{N}{100} \leqq \Phi \leqq \frac{N}{20}\right\}, \quad \left\{\frac{N}{100} \leqq \delta^2 \leqq \frac{N}{50}\right\}, \quad \left\{\frac{N}{100} \leqq \delta^3 \leqq \frac{N}{50}\right\},$$

respectively. Note that for $N \geqq 100$, $K_2K_3 \leqq K_1$, and that Φ in invariant under translations. It is easy to see that

$$\liminf_{n \to \infty} E(K_2K_3) \geqq c_1. \tag{3.1}$$

Let

$$\underset{\sim}{G}(\omega_1) = \inf_{-\infty < k < \infty} G^k(\omega_1)$$

where $G^k(\omega_1) = \infty$ if $A(k, \omega_1) = \infty$ and otherwise

$$G^k(\omega_1) = \sum_{m=-\infty}^{\infty} g(A(m+k, \omega_1) - A(k, \omega_1)),$$

where g is as in Theorem 2.2. Note that $\underset{\sim}{G}$ is invariant under translations and hence so is $\underset{\sim}{G}K_1$. By Corolllary 2.5, for any $a \in \mathbf{R}$

$$E_1(\underset{\sim}{G}JD^+I\{\underset{\sim}{G} = a\}K_1) = E_1(K_1I\{\underset{\sim}{G} = a\}).$$

But $G \geqq a$ on the set $\{\underset{\sim}{G} = a\}$, hence

$$E_1(JD^+I\{\underset{\sim}{G} = a\}K_1) \leqq \frac{1}{a} E_1(K_1I\{\underset{\sim}{G} = a\}),$$

or, by integrating on a (since there are only a countable number of finite random walk paths, $\underset{\sim}{G}$ takes on only a discrete number of values so we are really summing),

$$E_1(JD^+K_1) \leqq E_1((\underset{\sim}{G})^{-1}K_1). \tag{3.2}$$

A standard estimate using the local central limit theorem (see e.g. Spitzer [10]) gives

$$\sum_{n=0}^{|x|^2} P\{S^2(n) = x\} \geqq c_1(|x|^{-(d-2)} \wedge 1).$$

Since a random walk with killing rate $\beta = 1/N$ has probability $(1-\beta)^{|x|^2}$ of taking at least $|x|^2$ steps we get

$$\begin{aligned} g(x) &\geqq c_1(1-\beta)^{|x|^2}(|x|^{-(d-2)} \wedge 1) \\ &\geqq c_1 h(x) \end{aligned}$$

where

$$h(x) = e^{-|x|^2/N}(|x|^{-(d-2)} \wedge 1).$$

If we let

$$S(n, \omega_1) = \begin{cases} S^2(n, \omega_1), & n \geqq 0, \\ S^3(-n, \omega_1), & n \leqq 0, \end{cases}$$

then if $N/100 \leqq \Phi(\omega_1) \leqq N/20$,

$$\underset{\sim}{G}(\omega_1) \geqq \frac{N}{100} \inf_{-N/20 \leqq i,j \leqq N/20} g(S(i, \omega_1) - S(j, \omega_1)).$$

Hence if we define

$$\rho(\omega_1) = \sup_{-N/20 \leqq i,j \leqq N/20} |S(i, \omega_1) - S(j, \omega_1)|,$$

$$H(\omega_1) = \inf_{-N/20 \leqq i,j \leqq N/20} h(S(i, \omega_1) - S(j, \omega_1)),$$

then

$$E_1(G^{-1}K_1) \leqq c_2 N^{-1} E_1(H^{-1}) \leqq c_2 N^{-1} E_1(e^{-\rho^2/N}\rho^{d-2}). \tag{3.3}$$

But the central limit theorem and reflection principle give for $r > 0$

$$\begin{aligned} P\{\rho \geqq r\sqrt{N}\} &\leqq P\left\{\sup_{|i| \leqq N/20} \left|S(i) - S\left(\frac{N}{20}\right)\right| \geqq r\,\frac{\sqrt{N}}{2}\right\} \\ &= P\left\{\sup_{0 \leqq i \leqq N/10} |S(i)| \geqq r\,\frac{\sqrt{N}}{2}\right\} \\ &\leqq 2P\left\{\left|S\left(\frac{N}{10}\right)\right| \geqq r\,\frac{\sqrt{N}}{2}\right\} \leqq c_2 \exp\left\{-\frac{5r^2}{4\lambda}\right\}. \end{aligned}$$

Hence

$$E_1(e^{-\rho^2/N}\rho^{d-2}) \leqq c_2 N^{(d-2)/2}.$$

Combining this with (3.2), (3.3) and the estimate $K_2K_3 \leqq K_1$, we get

$$E_1(JD^+K_2K_3) \leqq \begin{cases} c_2 N^{-1/2}, & d = 3, \\ c_2 N^{-1}, & d = 2. \end{cases} \tag{3.4}$$

By (3.1),

$$\begin{aligned} E_1(JD^+K_2K_3) &= P\{K_2K_3 = 1\} E_1(JD^+ \mid K_2K_3 = 1) \\ &\geqq c_1 E_1(\tilde{J}_{1/50} D_{1/50}) \end{aligned}$$

where

$$\tilde{J}_\alpha(\omega_1) = P_2\{S^1(j, \omega_2) \neq S(k, \omega_1), -N \leqq k \leqq N, 0 < j \leqq \alpha N\},$$

$$D_\alpha(\omega_1) = I\{S^2(k, \omega_1) \neq 0, 0 < k \leqq \alpha N\}.$$

Therefore, (3.4) implies

$$E_1(\tilde{J}D) \leqq \begin{cases} c_2 N^{-1/2}, & d = 3, \\ c_2 N^{-1}, & d = 2, \end{cases} \tag{3.5}$$

where $\tilde{J} = \tilde{J}_1$, $D = D_1$. It remains to prove the estimate for $E_1(\tilde{J}) = F(N)$.

Let $\sigma_0 = 0$ and for $i > 0$,

$$\sigma_i = \inf\{m > \sigma_{i-1} : S^2(m) = 0\}.$$

Let

$$\Delta = \sup\{n : \sigma_n \leqq N\},$$

and let B_n, V_m be the indicator functions of the events $\{\Delta = n\}$, $\{\sigma_\Delta = m\}$, respectively. Let L_n be the indicator function of the event

$$\{\exists e, |e| = 1, S^2(j, \omega_1) \neq e, \text{ for } j = \sigma_r + 1, 0 \leqq r < n\}.$$

Since returns to the origin are independent, and $S(\sigma_r + 1)$ is a point of norm one,

$$E(L_n) \leqq \left(\frac{2d-1}{2d}\right)^{n-1}. \tag{3.6}$$

Let

$$\hat{J}_m(\omega_1) = P_2\{S^1(j, \omega_2) \neq S(k, \omega_1), 0 < j \leqq N, \; -N \leqq k \leqq 0 \text{ or } N - m < k \leqq N\},$$

$$\hat{D}_m(\omega_1) = I\{S(k, \omega_1) \neq 0, N - m < k \leqq N\}.$$

Then $B_n V_m = I\{\sigma_n = m\}\hat{D}_m$. Clearly if $A \subset \mathbf{Z}^d$ contains all the points of norm one, then a random walk starting at 0 hits A. Also, given $S(0) = S(m) = 0$, $\{S(j) : 0 \leqq j \leqq m\}$ and $\{S(j) : -N \leqq j \leqq 0, m \leqq j \leqq N\}$ are independent. We then get, using (3.5) and (3.6),

$$\begin{aligned} E_1(\tilde{J} B_n V_m) &\leqq E_1(\tilde{J} I\{\sigma_n = m\}\hat{D}_m) \\ &\leqq E_1(I\{\sigma_n = m\} L_n \hat{J}_m \hat{D}_m). \\ &\leqq c_2 P_1\{\sigma_n = m\}(N-m)^{(d-4)/2}\left(\frac{2d-1}{2d}\right)^{n-1}. \end{aligned} \tag{3.7}$$

For $d = 3$, we need only the standard estimate

$$\begin{aligned} P_1\{\sigma_\Delta \geqq N/2\} &\leqq P_1\{S(j) = 0 \text{ for some } j \geqq N/2\} \\ &\leqq c_2 N^{-1/2}, \end{aligned}$$

to conclude

$$\begin{aligned} E_1(\tilde{J}) &= \sum_{m=0}^{N} \sum_{n=0}^{N} E_1(\tilde{J} B_n V_m) \\ &\leqq c_2 (N/2)^{-1/2} P_1\{\sigma_\Delta < N/2\} + P\{\sigma_\Delta \geqq N/2\} \\ &\leqq c_2 N^{-1/2}. \end{aligned}$$

For $d = 2$ we need the estimate

$$P\{\sigma_n = m\} \leqq \frac{c_2 n^2}{m(\log m)^2}, \qquad m \neq 0, \tag{3.8}$$

which we prove below. Using (3.8), we get

$$\begin{aligned} E_1(\tilde{J}) &\leqq c_2 \sum_{m=0}^{N} \sum_{n=0}^{N} P\{\sigma_n = m\} \left(\frac{3}{4}\right)^{n-1} \{(N-m)^{-1} \wedge 1\} \\ &\leqq P\{\sigma_\Delta = N\} + c_2 \sum_{m=1}^{N-1} \sum_{n=0}^{N} \frac{n^2}{m(\log m)^2} \left(\frac{3}{4}\right)^{n-1} \frac{1}{N-m} \\ &\quad + c_2 \sum_{n=0}^{\infty} P\{\sigma_n = 0\} \left(\frac{3}{4}\right)^{n-1} \frac{1}{N} \\ &\leqq c_2 N^{-1} + c_2 \sum_{m=1}^{N-1} \frac{1}{m(\log m)^2 (N-m)} + c_2 N^{-1} \\ &\leqq c_2 N^{-} 1. \end{aligned}$$

To prove (3.8), let $X_i = \sigma_i - \sigma_{i-1}$. Then $X_1, X_2, \ldots, X_n$ are i.i.d. random variables and $P\{\sigma_n = m\} = P\{X_1 + \cdots + X_n = m\}$. We first derive the estimate

$$P\{X_1 = k\} \leqq c_2 k^{-1} (\log k)^{-1}. \tag{3.9}$$

To prove this consider $q(j, x) = P\{S(j) = x; S(i) \neq 0, 0 < i \leqq j/2\}$. Then by a standard estimate and the Local Central Limit Theorem,

$$\begin{aligned} q(j, x) &\leqq P\{S(i) \neq 0, 0 < i \leqq j/2\} \sup_y P\{S(j) = x \mid S([j/2]) = y\} \\ &\leqq c_2 (\log k)^{-1} k^{-1}. \end{aligned}$$

For k even, by splitting the path into two pieces and reversing time,

$$P\{X_1 = k\} = \sum_x \left(q\left(\frac{k}{2}, x\right)\right)^2.$$

But since

$$\sum_x q\left(\frac{k}{2}, x\right) \leqq c_2 (\log k)^{-1} \quad \text{and} \quad q\left(\frac{k}{2}, x\right) \leqq c_2 (\log k)^{-1} k^{-1},$$

a simple argument gives (3.9). The inequality for (3.8) is obvious if $n \geqq m^{3/4}$, so we may assume $n \leqq m^{3/4}$. Then,

$$
\begin{aligned}
&P\{X_1 + \cdots + X_n = m\} \\
&\leqq \sum_{j=1}^{n} P\{X_1 + \cdots + X_n = m, X_j \geqq m/n\} \\
&= nP\{X_1 + \cdots + X_n = m, X_1 \geqq m/n\} \\
&\leqq nP\{X_1 + \cdots + X_n = m \mid X_2 + \cdots + X_n \leqq m - m/n\} \\
&\leqq c_2 n^2 m^{-1} (\log m)^{-2}.
\end{aligned}
$$

4. Lower bound for $F(N)$

The main result needed for the lower bound is the following proposition which essentially states that a random walk path for $d = 2, 3$ has positive capacity. Proofs can be found in Erdös and Taylor [3] or Felder and Fröhlich [4].

LEMMA 4.1. *There exists a $c_1 > 0$ such that if S and S^2 are independent random walks starting of 0 and x respectively with $|x| \leqq 2N^{1/2}$ and*

$$\tilde{C} = \{S(-N, N) \cap S^2(0, N) \neq \varnothing\},$$

then

$$P(\tilde{C}) \geqq c_1.$$

We need a slight improvement on this.

LEMMA 4.2. *Let*

$$A = \{S(-3N, 3N) \cap S^2(0, N) = \varnothing\},$$

$$B = \{(S(-3N, -2N) \cup S(2N, 3N)) \cap S^2(0, 2N) = \varnothing\},$$

$$C = \{S(-N, N) \cap S^2(0, 2N) \neq \varnothing\}.$$

Then there exists a constant $c_1 > 0$ such that if $\frac{1}{2}N^{1/2} \leqq |x| \leqq N^{1/2}$,

$$\liminf_{N \to \infty} P(A \cap B \cap C) \geqq c_1.$$

PROOF. Let

$$D_r = \{|S(j) - x| \geqq 3r N^{1/2}, -N \leqq j \leqq N\}.$$

Then

$$\lim_{r\to 0} \liminf_{N\to\infty} P(D_r) = 1.$$

(This can be seen from the invariance principle and the fact that Brownian motion does not hit points for $d \geqq 2$.) Hence by Lemma 4.1 there is an $r > 0$ such that

$$\liminf_{N\to\infty} \left(\inf_{|y-x| \leqq rN^{1/2}} P^{0,y}(D_r \cap \{S(-N, N) \cap S^2(0, N) \neq \varnothing\}) \right) \geqq c_1,$$

where $P^{0,y}$ indicates probabilities assuming $S(0) = 0$ and $S^2(0) = y$. If

$$L_r = \{ |S^2(j) - x| \leqq rN^{1/2}, 0 \leqq j \leqq N \},$$

$P(L_r) \geqq c_1$ and by a standard Markov argument we get

$$\liminf_{N\to\infty} P(L_r \cap C \cap D_r) \geqq c_1.$$

If

$$M_R = \{ |S^2(j)| \leqq RN^{1/2}, 0 \leqq j \leqq 2N \}$$

then

$$\lim_{R\to\infty} \liminf_{N\to\infty} P(M_R) = 1$$

and hence for some $R < \infty$

$$\liminf_{N\to\infty} P(L_r \cap C \cap D_r \cap M_R) \geqq c_1.$$

Finally, if

$$Q = \{ |S(j)| \geqq 2RN^{1/2}, 2N \leqq |j| \leqq 3N; \ |S(j) - x| \geqq 2rN^{1/2}, N \leqq |j| \leqq 3N \},$$

then

$$E(I_Q \mid S(j), -N \leqq j \leqq N) \geqq c_1$$

on D_r, and hence

$$P(L_r \cap C \cap D_r \cap M_R \cap Q) \geqq c_1.$$

But $A \cap B \cap C \supset L_r \cap C \cap D_r \cap M_R \cap Q$ so the lemma is proved.

Proof of the Lower Bound. If $S(0) = 0$, $S^2(0) = x$ let

$$\tau = \inf\{ j \geqq 0 : S^2(j) \in S(-3N, 3N) \}, \qquad \sigma = \sup\{ k \leqq 3N : S(k) = S^2(\tau) \}.$$

Then Lemma 4.2 gives that

$$(4.1)\qquad \sum_{x\in\mathbf{Z}^d}\left(\sum_{j=N}^{2N}\sum_{k=-2N}^{2N} P^{0,x}\{\tau=j,\sigma=k\}\right)\geqq c_1N^{d/2}.$$

But for a fixed j, k, with $N\leqq j\leqq 2N$, $-2N\leqq k\leqq 2N$, by reversing paths we can see that

$$\sum_{x\in\mathbf{Z}^d} P^{0,x}\{\tau=j,\sigma=k\}\leqq F(N),$$

and hence

$$(4.2)\qquad \sum_{k=-2N}^{2N}\sum_{j=N}^{2N}\sum_{x\in\mathbf{Z}^d} P^{0,x}\{\tau=j,\sigma=k\}\leqq c_2N^2F(N).$$

The lower bound follows from (4.1) and (4.2).

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